ON SYSTEMS OF SECOND-ORDER VARIATIONAL INEQUALITIES†

BY

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ABSTRACT

Let u be a solution of an elliptic (linear or nonlinear) variational inequality with obstacle. Under natural smoothness conditions put upon the data, it is shown that the second derivatives of u lie in a certain Morrey space and hence, in the case of two independent variables, the solution u has a Hölder continuous gradient.

In this paper, we consider the problem of interior regularity of solutions of systems of second-order elliptic variational inequalities with obstacles. Under natural smoothness assumptions on the data, it is shown that the second derivatives of any solution lie in certain Morrey spaces and hence, in the case of *two* independent variables, the first derivatives are Hölder continuous. It is well known that one cannot expect the second derivatives to be Hölder continuous. We treat the linear and the nonlinear case.

For *single* second order variational inequalities (i.e. not systems) and for second-order systems in *one* variable, the Hölder continuity of the gradient of the solution is well known (see Lewy and Stampacchia [7,8], Brezis and Stampacchia [1], Schiaffino and Troianello [12]). For systems in more than one independent variable, regularity results have been obtained by Tomi [13, 14] and Hildebrandt [5]. Tomi and Hildebrandt treat a rather general side condition, but neither author considers systems with a general elliptic principal part, instead, they assume some separating condition. A similar condition was also assumed

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by Vergara Caffarelli [15], who studied systems of variational inequalities with the side condition corresponding to the case of two elasitc membranes, one lying above the other.

NOTATION

Ω = bounded open subset of the n-dimensional euclidean space R^n, $n=2, 3, \cdots$

$L^p(\Omega)$ (shorter: L^p) = Lebesgue space on Ω with norm $\|u\|_p = (\int |u|^p\,dx)^{1/p}$, $1 \leqq p < \infty$, $\|u\|_\infty$ = ess sup $\{|u(x)|,\ x \in \Omega\}$

$\int$ = integration over Ω in the sense of Lebesgue

$H^{m,p}(\Omega)$ (shorter: $H^{m,p}$) = Sobolev space on Ω with norm $\|u\|_{m,p} = \Sigma_k \|\nabla_k u\|_p$, $(k = 0, \cdots, m)$

$H^m = H^{m,2}$

∇^k = vector of generalized derivatives of u of order k

∂_i = derivative with respect to the i-th argument

∂_0 = identity

$C(B)$, $C^1(B)$, $C^\alpha(B)$ resp. $C^{1+\alpha}(B)$ = space of real functions on the open subset $B \subset R^n$ which are continuous, continuously differentiable, Hölder continuous with exponent $\alpha \in (0,1)$, resp. Hölder continuously differentiable in B

$C_0^\infty(\Omega)$ = space of test functions on Ω

$H_0^1(\Omega)$ = closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$

$H_{loc}^{m,p}$ = space of functions on Ω whose restrictions are in $H^{m,p}(\Omega_0)$ for every $\Omega_0 \subset \subset \Omega$ (i.e. Ω_0 compactly contained in Ω)

$L_{loc}^{2,\beta}(\Omega)$ = space of L^2-functions on Ω which satisfy a local Morrey condition, i.e. for every $u \in L_{loc}^{2,\beta}$ and every $\Omega_0 \subset \subset \Omega$, there exists a constant K such that $\int_R u^2 dx \leqq KR^{2\cdot\beta}$ $\beta \in (0,1)$, for every ball B_R of radius R contained in Ω_0, where $\int_R$ denotes the integration over B_R. Finally, let $[W]^r$ denote the space of r-vector valued functions with components in W, where W is one of the function spaces defined above. The corresponding norms are denoted as in the case $r = 1$.

The variational inequality. Let V be a closed subspace of $[H^1]^r$ containing $[H_0^1]$, and let F_i, $i=0,\cdots,n$, resp. ψ r-vector-valued real functions on $\Omega \times R^{r(n+1)}$ resp. Ω for which some smoothness assumptions will be made.

Let $\mathbf{K} = \{w \in V \mid w \geqq \psi \text{ in } [H^1]^r\}$. (For the definition "$w \geqq \psi$ in H^1", see [7].) The problem is:

Find $u \in \mathbf{K}$ *such that*

(1) $\langle Tu, u-v\rangle := \Sigma_i \int F_i(x,u,\nabla u)\cdot\partial_i(u-v)dx \leqq 0$, $(i = 0,\cdots,n)$, for all $v \in \mathbf{K}$.

The point $\cdot$ denotes the r-dimensional euclidean scalar product.

It is well known that problem (1) has a solution if the mapping $T: V \to V$ defined in (1) is **K**-coercive and of type L in the sense of [1]. This can be assured by some growth, coerciveness, and monotonicity assumptions on the coefficients F_i (see [1], [6], and [10, Chapter 5. 12]). In this connection, we remark that the additional coerciveness hypothesis in [6] for the principal part of the nonlinear differential operator is not necessary (see [4]).

For our regularity theorem, we need the following assumptions for the coefficients F_i:

(A) *The case with nonlinear principal part.*

I. *Smoothness.* Let $F_i : \Omega \times R^{r(n+1)} \to R^r$ be functions such that

i) $F_i \in [C^1(\Omega \times R^{r(n+1)})]^r$, $i = 1, \cdots, n$

ii) $F_0(x,.) \in [C(R^{r(n+1)})]^r$ for almost all $x \in \Omega$

iii) F_0 is measurable on $\Omega \times R^{r(n+1)}$.

II. *Growth condition.* Let F_{ix} and $F_{i\eta}$ denote the vectors of the first partial derivatives of all the components of $F_i(x, \eta)$ with respect to x resp. to η. We assume there exist constants K, Λ, β, $0 < \beta < 1$, and functions $g \in L^2(\Omega)$, $g_0 \in L^{2,\beta}_{\mathrm{loc}}(\Omega) \cap L^2$ such that for all $\eta \in R^{r(n+1)}$ and almost all x the following inequalities hold:

i) $|F_i(x, \eta)| \leqq K|\eta| + g(x)$, $i = 1, \cdots, n$,

ii) $|F_{ix}(x,\eta)| \leqq K|\eta| + g_0(x)$, $i = 1, \cdots, n$,

iii) $|F_{i\eta}(x,\eta)| \leqq K + \; g_0(x)$, $i = 1, \cdots, n$,

iv) $|F_0(x,\eta)| \leqq K|\eta| + g_0(z)$.

III. *Strong uniform ellipticity.* Let $F^{\nu}_{i,k\mu}$ denote the partial derivative of the ν-th component of F_i with respect to the argument of F_i in (1) which corresponds to $\partial_k u_\mu$ where u_μ is the μ-th component of u. We assume there exists a constant $\lambda > 0$ such that

$$\Sigma^{\nu\mu}_{ik} F^{\nu}_{i,k\mu} \xi_{i\nu} \xi_{k\mu} \geqq \lambda |\xi|^2, \; (i, k = 1, \cdots, n; \; \nu, \mu = 1, \cdots, r),$$

for all $\xi \in R^{rn}$ with components $\xi_{i\nu}$.

(B) *Conditions in the case with linear principal part.*

In this case, we assume that $F_i(x,\eta)$ is linear in η, i.e.

$$F_i(x,\eta) = \; \Sigma_{k\mu} a^{\nu\mu}_{ik}(x)\eta_{\mu k} + f^{\nu}_i(x), \; k = 0, \cdots, n; \mu = 1, \cdots, r; \; i = 1, \cdots, n; \nu = 1, \cdots, r$$

where the a_{ik} and f^{ν}_i satisfy the following conditions:

I′, II′. *Smoothness and growth conditions.*

i′) $a^{\nu\mu}_{ik} \in C^1(\Omega) \cap L^\infty(\Omega)$

ii′) $f_i^\nu \in L^2(\Omega) \cap H^1_{loc}(\Omega)$

iii′) $\nabla f_i^\nu \in L^{2,\beta}_{loc}(\Omega)$

for $i = 1, \cdots, n$; $k = 0, \cdots, n$; $\nu, \mu = 1, \cdots, r$.

III′. *Ellipticity.* There exists a constant $\lambda > 0$ such that $\sum_{ik}^{\nu\mu} a_{ik}^{\nu\mu} \zeta_i \zeta_k \pi_\nu \pi_\mu \geqq \lambda |\zeta|^2 |\pi|^2$, $(i, k = 1, \cdots, n;\ \nu, \mu = 1, \cdots, r)$, for $\zeta \in R^n$ and $\pi \in R^r$.

The conditions for F_0 are the same as in the nonlinear case (A).

With these assumptions, we can obtain the following

THEOREM. *Let the coefficients F_i in* (1) *satisfy the conditions* A *or* B *and the obstacle ψ the condition $\nabla^2 \psi \in [L^{2,\beta}_{loc}(\Omega)]^{n \cdot n \cdot r}$. Then every solution $u \in \mathbf{K}$ of* (1) *has second derivatives $\nabla^2 u \in [L^{2,\alpha}_{loc}(\Omega)]^{n \cdot n \cdot r}$ for some $\alpha \in (0,1)$.*

COROLLARY. *If $n = 2$, then $u \in [C^{1+\alpha}(\Omega)]^r$.*

PROOF OF THE THEOREM. The first step consists in proving $u \in [H^2_{loc}(\Omega)]^r$. This follows by setting

(2)
$$v_\varepsilon = u + \varepsilon D_{-ih}(\phi^2 D_{ih}(u - \psi))$$

where $\phi \in C^\infty(\Omega)$, $\phi \geqq 0$, and $D_{\pm ih} z(x) = \pm h^{-1}(z(x \pm h e_i) - z(x))$, $h > 0$, with e_i being the *i-th* unit vector. If h and $\varepsilon = \varepsilon(h)$ are small enough, it is easy to see that $v_\varepsilon \in \mathbf{K}$. Inserting $v = v_\varepsilon$ into the variational inequality, one obtains by classical techinques of the theory of elliptic equations an estimate

$$\| \phi \nabla D_{ih} u \|_2 \leqq C \text{ uniformly for } h \to 0$$

and all admissible ϕ. From this the desired fact follows. This method was used by Lions [9]. Another method was used by the author in [3] where it was shown that

$$v_\varepsilon := u + \varepsilon \phi^2 D_{-ih} D_{ih}(g \cdot (u + a)) \in \mathbf{K}$$

for a certain auxiliary function $g > 0$ and a certain number a. From this the differentiability $u \in [H^2_{loc}]^r$ follows too. This method has the advantage that one can also treat the case of *two* obstacles, i.e. $\psi_1 \leqq u \leqq \psi_2$ (which are allowed to have convex resp. concave corners).

The second step consists of proving a Morrey condition for $\nabla^2 u$. For every closed ball $B_R \subset \Omega$ of radius R with the property that the concentric ball B_{2R} with radius $2R$ is contained in Ω, we construct a function $\phi \in H^{2,\infty}(\Omega)$ such that

$$\phi = 1 \text{ on } B_R,\ \phi = 0 \text{ on } \Omega - B_{2R},\ 0 \leqq \phi \leqq 1 \text{ a.e. in } \Omega, \text{ and}$$

$$|\nabla\phi| \leqq K'R^{-1},\ |\nabla^2\phi| \leqq K'R^{-2} \text{ in } B_{2R}-B_R \tag{3}$$

with some constant K'. Such a function ϕ can easily be constructed. Choosing the number $h > 0$ smaller than dist $(\partial B_{2R}, \partial\Omega)$ and $\varepsilon \in (0, h^2]$, one has for this ϕ

$$u_\varepsilon := u + \varepsilon\phi^2 D_{-jh}D_{jh}(u - \psi) \in \mathbf{K},\ j \in \{1, \cdots, n\}. \tag{4}$$

In fact, since ϕ has compact support and $V \in [H_0^1]^r$, it follows that $u_\varepsilon \in V$, and from $u \geqq \psi$ in $[H^1]^r$ one concludes (4) in an elementary way.

Thus, we may insert u_ε into the variational inequality (1) and obtain

$$- \Sigma_i \int F_i \cdot \partial_i(\phi^2 D_{-jh}D_{jh}(u - \psi))dx \leqq 0, \quad (i = 0, \cdots, n). \tag{5}$$

Note that we may write

$$D_{-jh}D_{jh}(u - \psi) = D_{-jh}(- b_j + D_{jh}(u - \psi)) \tag{6}$$

and in (4), we may replace F_i by

$$F_i - d_i, \quad i = 1, \cdots, n, \tag{7}$$

since ϕ^2 has compact support in Ω. The vectors b_j, $d_i \in R^r$ will be defined later. In (4), we do some elementary calculations which are based on the identity

$$\begin{aligned}\int f \cdot \partial_i(\phi^2 D_{-hj}g)dx &= \int f \cdot \partial_i\phi^2 D_{-hj}g \ dx + \int f \cdot \phi^2 D_{-hj}\partial_i g dx \\ &= - \int D_{hj}(f\ \partial_i\phi^2) \cdot g dx - \int D_{hj}(f\phi^2) \cdot \partial_i g dx\end{aligned}$$

provided the terms have meaning. This identity can be applied to (5) with

$$f = F_i - d_i \text{ and } g = b_j + D_{hj}(u - \psi)$$

if we use remark (6) and (7). Going to the limit $h \to 0$, we arrive at the inequality

$$\begin{aligned}\Sigma_i \int [\partial_j((F_i - d_i)\partial_i\phi^2) \cdot (\partial_j(u - \psi) - b_j) + \partial_j((F_i - d_i)\phi^2) \cdot \partial_i\partial_j(u - \psi)]dx \\ - \int F_0\phi^2\partial_j^2(u - \psi)dx \leqq 0,\ (i = 1, \cdots, n).\end{aligned} \tag{8}$$

Using Hölder's inequality and the properties of ϕ stated in (3), we conclude from (8)

$$\Sigma_i \int \phi^2\partial_j F_i \cdot \partial_i\partial_j u dx \leqq \int_{2R0} (A + B + C)dx + \int_{2R} (D + E + F)dx, \quad (i = 1, \cdots, n), \tag{9}$$

where

$A = K(\varepsilon)R^{-2}\,\Sigma_i|F_i - d_i|^2,\ B = K(\varepsilon)\,\Sigma_i|\partial_j F_i|^2,\ (i = 1,\cdots,n),$

$C = K_1R^{-2}|\partial_j(u-\psi) - b_j|^2,\ D = \varepsilon|\partial_j\nabla u|^2,\ E = K(\varepsilon)|F_0|^2,\ F = K_1|\partial_j\nabla\psi|^2.$

The symbol $\int_{2R0}$ denotes the integration over the set $B_{2R} - B_R$: we used the fact that $|\nabla\phi^2| = |\nabla^2\phi^2| = 0$ on the complement of $B_{2R} - B_R$. The number $\varepsilon > 0$ is given and will be defined later. $K(\varepsilon)$ and K_1 are constants not depending on R. Now we set b_j resp. $d_i \in R^r$ equal to the mean values of $\partial_j(u-\psi)$ resp. F_i over $B_{2R} - B_R$. Then by the inhomogeneous Poincaré inequality, we may estimate

$$\int_{2R0} A\,dx \leqq \Sigma_i K_2 K(\varepsilon)\int_{2R0}|\nabla F_i|^2 dx,\ (i = 1,\cdots,n) \tag{10}$$

$$\int_{2R0} C\,dx \leqq K_2\int_{2R0}|\nabla\partial_j(u-\psi)|^2 dx. \tag{11}$$

Let us assume that $B_{2R} \subset \Omega' \subset\subset \Omega$. Calculating $\nabla F_i = \nabla F_i(.,u,\nabla u)$ and applying Condition II resp. I′, II′ , we we obtain

$$\int_{2R0}|\nabla F_i|^2 dx \leqq K_3\int_{2R0}|\nabla^2 u|^2\,dx + k_3R^\gamma. \tag{12}$$

The exponent $\gamma\in(0,1)$ depends on the exponent β of the hypothesis II resp. I′, II′, and on the exponent σ of the Morrey condition for ∇u which holds because of Sobolev's theorem. Furthermore, again because of Sobolev's theorem and Condition II (iv),

$$\int_{2R}|F_0|^2 dx \leqq K_3R^\gamma. \tag{13}$$

Recalling that we supposed a Morrey condition for $\nabla^2\psi$ and using (10), (11), (12), and (13), we may estimate the right-hand side of (9) by

$$\varepsilon\int_{2R}|\nabla^2 u|^2 dx + K'(\varepsilon)\int_{2R0}|\nabla^2 u|^2 dx + K_4R^\gamma. \tag{14}$$

Calculating $\partial_j F_i$ in the right-hand side of (9) we may estimate the lower order terms by an expression (14) using the same arguments as before. Thus, we arrive at the inequality

$$\Sigma_{ik}^{\nu\mu}\int\phi 2F_{ik}^{\nu\mu}\partial_k\partial_j u_\nu\partial_i\partial_j u_\mu dx \leqq \varepsilon\int_{2R}|\nabla^2 u|^2 dx + K'(\varepsilon)\int_{2R0}|\nabla^2 u|^2 dx + \tag{15}$$

$$+ K_4R^\gamma \quad (i,k = 1,\cdots,n;\ \nu,\mu = 1,\cdots,r).$$

In the nonlinear case, we may apply the ellipticity condition III. Summing from $j = 1, \cdots, n$, afterwards, we obtain

$$\int_R |\nabla^2 u|^2 dx \leqq \Gamma\varepsilon\int_{2R} |\nabla^2 u|^2 dx + \Gamma K'(\varepsilon)\int_{2R0} |\nabla^2 u|^2 dx + \Gamma K_4 R^\gamma \tag{16}$$

with some constant $\Gamma \sim \lambda^{-1}$.

In the linear case, we have a weaker ellipticity condition. In order to obtain (16) also in this case, we rewrite

$$\phi^2 F_{ik}^{\nu\mu}\partial_k\partial_j u_\nu \partial_i\partial_j u_\mu = F_{ik}\partial_k(\phi\cdot(\partial_j u_\nu - c_{j\nu}))\,\partial_i(\phi\cdot(\partial_j u_\mu - c_{j\mu})) + G \tag{17}$$

where

$$\Big|\int G dx\Big| \leqq K_5 R^{-2}\int_{2R0} |\partial_j u - c_j|^2 dx + K_5\int_{2R0} |\nabla^2 u|^2 dx.$$

Note that $F_{ik}^{\nu\mu} = a_{ik}^{\nu\mu}, i,k = 1, \cdots, n$, in the linear case.

On account of the ellipticity condition III′ and the smoothness condition I′, II′ (i′), we may estimate the form

$$Q = \int \Sigma_{ik}^{\nu\mu} F_{ik}^{\nu\mu}\partial_k v_\mu \partial_i dx, \quad (i,k = 1, \cdots, n\,;\ \nu, \mu = 1, \cdots, r)$$

from below by Garding's inequality for vector functions (see [11]):

$$Q \geqq \tfrac{1}{2}\lambda\|\nabla v\|_2^2 - C_0\|v\|_2^2,$$

where $v \in [H^1]^r$ has compact support, λ is the ellipticity constant and C_0 some other constant. Applying this to (15) we obtain in view of (17) that the left-hand side of (15) is larger than

$$L := \tfrac{1}{2}\lambda\|\nabla(\phi\cdot(\partial_j u - c_j))\|_2^2 - C_0|\phi(\partial_j u)\|_2^2 - C\Big|\int Cdx\Big|. \tag{18}$$

By an elementary calculation

$$L \geqq \tfrac{1}{2}\lambda\|\phi\cdot\nabla\partial_j u\|_2^2 - C'R^{-2}\int_{2R0} |\partial_j u - c_j|^2 dx - C_0\|\phi(\partial_j u - c_j)\|_2^2$$
$$-C'\int_{2R0} |\nabla^2 u|^2 dx.$$

Setting $x_j \in R^r$ equal to the mean value of $\partial_j u$ taken over $B_{2R} - B_R$ and applying the inhomogeneous Poincaré inequality, we obtain

$$R^{-2}\int_{2R0} |\partial_j u - c_j|^2 dx \leqq C''\int_{2R0} |\nabla\partial_j u|^2 dx. \tag{19}$$

By Sobolev's theorem

$$\| \phi \cdot (\partial_j u - c_j) \|_2^2 \leqq C_1 R^\sigma \tag{20}$$

for some $\sigma \in (0,1)$ since supp $\phi \subseteq B_{2R}$ and $u \in [H^2_{\mathrm{loc}}]^r$.

Finally, if we apply (18), (19), and (20) to (15), we arrive at an inequality of the type (16) also in the linear case.

Setting $\varepsilon = \frac{1}{2}\Gamma^{-1}$ in (15), we conclude

$$\int_R |\nabla^2 u|^2 dx \leqq K_6 \left(\int_{2R0} |\nabla^2 u|^2 dx + R^\gamma \right). \tag{21}$$

From the last inequality the desired Morrey condition follows by a technique which was used in [16]:

Adding the term $K_6 \int_R |\nabla^2 u|^2 dx$ to both parts of inequality (21) (i.e. filling up the "hole") and dividing by $1 + K_6$, we obtain

$$\int_R |\nabla^2 u|^2 dx \leqq \rho \int_{2R} |\nabla^2 u|^2 dx + \rho R^\gamma \tag{22}$$

where $\rho = K_6/(1 + K_6) < 1$. We choose $\alpha \in (0,1)$ such that $\rho 2^\alpha < 1$ and $\alpha \leqq \gamma$. If $R \leqq 1$, then $R^\gamma \leqq R^\alpha$. From (22), by iteration, it follows that

$$\int_r |\nabla^2 u|^2 dx \leqq \rho\, \Sigma_i (2^\alpha \rho)_i r^\alpha + \rho N^{-1} \int_R |\nabla^2 u|^2 dx, \ (i = 0, \cdots, N-1),$$

where $R = 2^{N-1} r$, provided $R \leqq 1$ and $B_R \subset \Omega$. Since

$$\rho\, \Sigma_i (2^\alpha \rho)^i < K_0 := \rho/(1 - 2^\alpha \rho), \ (i = 0, \cdots, N-1)$$

and $\rho < 2^{-\alpha}$, we conclude that

$$\int_r |\nabla^2 u|^2 dx \leqq K_0 r^\alpha + (r/R)^\alpha \int_R |\nabla^2 u|^2 dx$$

and

$$\int_r |\nabla^2 u|^2 dx \leqq k_0 r^\alpha + 2^\alpha (r/R)^\alpha \int_R |\nabla^2 u|^2 dx$$

for arbitrary $r \in (0, R/2)$, $R \leqq 1$, $B_R \subset \Omega$. Thus, the desired Morrey condition is derived and the theorem is proved. The corollary follows from Morrey's lemma (see [10], Th. 3.5.2.).

Added in proof

Note that we assumed a linear growth of the lower order term $F_0(x, u, \nabla u)$ with respect to ∇u. Hildebrandt [5] has shown that $u \in H^1$ implies $u \in H^{2,2}$ in the case of general two dimensional elliptic systems with quadratic growth for

$F_0(x,u,\nabla u)$ in ∇u. As a consequence, F_0 considered as a function depending only on x satisfies a Morrey condition and hence our condition iv). Thus also under the quadratic growth hypothesis, our theorem yields the step from $u \in H^{2,2}$ to $u \in C^{1+\alpha}$. The case of more general side conditions $u(x)K$ where K is a smooth subset of R^2 can be reduced to the case treated here by a local transformation see [14] or [5]. (For these reasons, we confined ourselves to the conditions presented here).

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